APPENDIX A

REMARKS ON THE JACOBIANS FOR THE GENERAL FLOW

Let's consider a general autonomous flow defined by the equation

$$\frac{d}{dt}x(t) = f(x(t)) \tag{149}$$

where $x \in \mathbb{R}^n$ and $f(.)$ is a smooth vector function. This flow will give an orbit $\phi(t,x)$ to each point x in the phase space,

$$\phi : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$$

which satisfies $\phi(0,x) = x$. For the invertible flow, $\phi(t,x)$ also satifies the full group property,

$$\phi(t2, \phi(t1,x)) = \phi(t2+t1,x)$$

Actually, in this case, $\phi(t,x)$ forms a one-parameter family of transformations. The Jacobian associated with an orbit is defined as

$$J^t(x) = \frac{\partial}{\partial x}\phi(t,x)$$

which describes the relative movement of the neighbouring points around the given orbit. Following the group property of $\phi(t,x)$, if $\phi(t_1,\tilde{x}) = x$, then by chain rule,

$$J^{t_1+t}(\tilde{x}) = J^t(x)J^{t_1}(\tilde{x}) \tag{150}$$

It is not hard to check that the Jacobian J^t satisfies the following equation,

$$\frac{d}{dx}J^t(x) = AJ^t$$

where $A(x) = \frac{\partial}{\partial x}f(x)$ is the gradient of the velocity field. If we have a periodic orbit of period T, then J^T is called monodromy matrix whose eigenvalues are independent of the starting point. Of course, the eigenvectors will be different for different starting points. However, they could be related by the natural evolution along the flow. Suppose that along the periodic orbit, there are two points $\tilde{x}, x$ with $\phi(t_1,\tilde{x}) = x$. If $\vec{\Lambda}$ is an eigenvector of $J^T(\tilde{x})$, i.e. $J^T(\tilde{x})\vec{\Lambda} = \lambda\vec{\Lambda}$, then $J^{t_1}(\tilde{x})\vec{\Lambda}$ is an eigenvector of $J^T(x)$ of the same eigenvalue as shown below by repeated use of (150),

$$J^T(x)(J^{t_1}(\tilde{x})\vec{\Lambda}) = J^{t_1}(\tilde{x})J^{T-t1}(x)J^{t_1}(\tilde{x})\vec{\Lambda} = J^{t_1}(\tilde{x})(J^T(\tilde{x})\vec{\Lambda}) = \lambda(J^{t_1}(\tilde{x})\vec{\Lambda})$$

In practical calculations, Poincare sections are frequently utilized and the flow itself is treated as a map on the Poincare section. Periodic orbits then become finite point sets and the corresponding Jacobians depend not only on the position but also the orientation of the Poincare section. The questions arise that whether the eigenvalues of the Jacobians are invariant and how they are related to those of the flow. We will see in the following

that the $n-1$ eigenvalues of the map are the same as the corresponding ones of the flow, independent of the choice of the Poincare section.

For simplicity, let's assume that the Poincare section is defined by $x_1 = 0$ and we have found a periodic orbit with the starting point on the Poincare section. The Jacobian in the full phase space is denoted by $J = (J_{.1}, \cdots, J_{.n})$, where $J_{.k}$ is the kth column of the Jacobian matrix. We denote it this way because it is the resulting point having evolved for one period of the neighbouring point in the kth direction around the periodic point on the Poincare section. Generally, $J_{1k} \neq 0$, which means that the final point is not on the Poincare section. If we bring it back along the flow to the Poincare section, then the corresponding Jacobian of the Poincare map will be constructed for the periodic orbit. The back-flying time t_k should satisfy $t_k v_1 = J_{1k}$ where v_1 is the first component of the velocity at the periodic point. Then $t_k = J_{1k}/v_1$ and $\tilde{J}_{.k} = J_{.k} - t_k\vec{v}$ would be the corresponding vector on the Poincare section. The Jacobian of the map could be written as

$$\tilde{J} = (\tilde{J}_{.1}, \cdots, \tilde{J}_{.n}) = J - \vec{v}\vec{t} \tag{151}$$

where $\vec{t} = col(t_1, \cdots, t_n)$ and $col(.)$ denotes the column vector. Now suppose that we have an eigenvector $\vec{\Lambda}$ of J, i.e. $J\vec{\Lambda} = \lambda\vec{\Lambda}$. We claim that $\tilde{\Lambda} = \vec{\Lambda} - t_\lambda\vec{v}$ is an eigenvector of $\tilde{J}$ of the same eigenvalue λ, where $t_\lambda = \frac{\Lambda_1}{v_1}$. In fact,

$$\begin{aligned}
\tilde{J}\tilde{\Lambda} &= (J - \vec{v}\vec{t})(\vec{\Lambda} - t_\lambda\vec{v}) \\
&= \lambda\vec{\Lambda} - t_\lambda\vec{v} - \vec{v}(\vec{t}\cdot\vec{\Lambda}) + t_\lambda\vec{v}(\vec{t}\cdot\vec{v}) \\
&= \lambda\vec{\Lambda} - \lambda t_\lambda\vec{v} = \lambda\tilde{\Lambda}
\end{aligned}$$

During the above calculation, we have used

$$\begin{aligned}
\vec{t}\cdot\vec{\Lambda} &= \sum_i t_i\Lambda_i = \sum_i \frac{J_{1i}}{v_1}\Lambda_i = \lambda\frac{\Lambda_1}{v_1} = \lambda t_\lambda \\
\vec{t}\cdot\vec{v} &= \frac{v_1}{v_1} = 1
\end{aligned}$$

since $J\vec{\Lambda} = \lambda\vec{\Lambda}$, $J\vec{v} = \vec{v}$. Note that $\tilde{v} = \vec{v} - \vec{v} = 0$, which will give the zero eigenvalue of $\tilde{J}$. In the full space, $\tilde{J}$ is still an $n \times n$ matrix with the first row filled with zeros. The reduced $(n-1)\times(n-1)$ matrix which is a restraint of $\tilde{J}$ on the Poincare section is given just by deleting the first row and column of $\tilde{J}$.
For the whole discussion above to be valid, $v_1 \neq 0$ should always hold, i.e. the flow should be transversal to the Poincare section. The reduced Jacobian is therefore the Jacobian in the transversal direction, the invariance of whose eigenvalues justifies the use of the notation J^T in Chapter 7.