CHAPTER I

INTRODUCTION

If I have seen less far than other men it is because I have stood behind giants.
E. A. Spiegel

In this chapter we give a sketch of dynamical system theory and turbulence study from both the historical development and methodology points of view. Since both subjects involve a huge body of literature, they are touched in a way that will provide background for and motivate our investigation later on. The bottomline is the effort that brings dynamical system theory, especially the periodic orbit theory, to the study of 1-d spatiotemporal chaos, a first step towards the real world 3-d turbulence.

1.1 *From dynamical systems to periodic orbit theory*

Dynamical system theory is a modern subject, but the basic elements emerged a century ago. Today, both its concepts and methodology are under intensive development. Periodic orbit theory is among the most powerful and promising of these developments. It combines many techniques for dealing with dynamical systems.

1.1.1 Historical perspective

Since the invention of calculus by Newton and Leibniz, differential equations become a focus of study, not only because of their dominant role in describing nature but also because of their intrinsic complexity and beauty and paramount importance in the development of new theories. Although more efforts are still being made to seek fundamental laws of nature in extremely high energy regime, physicists may proudly say that they already hold the laws necessary to explain and predict what happens in the physical world, except for events taking place immediately after the Big Bang or under extreme conditions that are far beyond the experience of the earthlings. Macroscopically, everything seems to be deterministic, just like what Leibniz pointed out in *Von dem Verhängnisse*: "Given knowledge of present state of the universe, we would see the future in the present as in a mirror."

This is an enticing remark and people have been investigating its validity ever since then. Mathematically, this is done by solving differential equations. The earliest attempts were devoted to obtaining explicit expressions in terms of known functions or quadratures. People had been devising various transformations and introducing and studying more and more special functions. Still, in many cases, they could not get what they wanted and tended to attribute the failure to their lack of cleverness in finding the appropriate transformation that could settle the problem.

The French mathematician Henry Poincaré broke the religion and gave new life to the old subject. In his study of celestial mechanics [205], Poincaré found that the equations of motion of heavenly bodies were in general not integrable, i.e. no smooth transformation existed to reduce the problem to a quadrature. In this sense, exact solutions of some equations are not possible. Instead, Poincaré suggested the qualitative method to study differential equations. An equation can be viewed as a dynamical system associated with a phase space, the orbit structure of which gives the qualitative properties of solutions of the equation. Geometrical and topological methods can then be invoked to study the phase space itself. This led Poincaré to his discovery of many important theorems in the theory of ordinary differential equations and chaos in celestial mechanics.

It was not until 1963 when Lorenz discovered chaotic strange attractor [164] in the atmospheric study that the chaotic dynamical systems started to attract attention of both mathematicians and scientists in general. Lorenz found that the solution of a simple 3-d dynamical system depends on the initial condition so sensitively that any small discrepancy will grow quickly to cover the whole attractor and make the system behavior totally unpredictable in the long run. This fact was later ascribed the term "butterfly effect". Researchers soon discovered that chaos is ubiquitous and accounts for various dynamical phenomena.

From then on, dynamical systems become a hot research area and its content has been extended from ordinary differential equations (ODEs) to partial differential equations (PDEs) [249, 114] and differentiable mappings [63, 42]. The development of digital computer boosted the research greatly with new fields opened to scientists like cellular automata [277] and lattice field theory [45]. Today, research on dynamical systems is rapidly expanding and rewarding in both theory and applications. It leads to many new concepts and methods but raises even more problems and conjectures awaiting an answer. First we review briefly the notions and techniques commonly used in dynamical sytem theory and later use these concepts to illuminate the way in our study of turbulence.

1.1.2 Different approaches to dynamical systems

Analytical manipulation is dear to most scientists for solving differential equations. If a differential equation is reducible to finitely many algebraic equations, even though the algebraic equations may not be explicitly solvable, the problem is considered to be solved since the global structure of solutions is known in principle and specific solutions can be calculated to any accuracy in the whole domain of definition [135, 138, 251, 124]. $n-1$ first integrals yield a full solution of an n-dimensional system. Although the direct integration or series expansion is most effective to linear systems, various technique have been devised to test the integrability of nonlinear equations, like the Painléve [135, 124, 274, 272, 44] and inverse scattering method [1]. Generally, integrability only exist in highly simplied models, but its discussion providing the starting point for studying more complex behavior.

The qualitative theory that combines algebra, analysis and geometry is the main tool to study the behavior of general differential equations. We care most about the long-term dynamics which concentrates on a small subset of the phase space in a dissipative system [202, 170]. To understand the asymptotic behavior of a system, we only need to identify the subset, and study the dynamics around it [107, 275, 111, 112]. The Banach's contraction mapping principle [113, 119, 230] plays a central role in discussing the existence and analytical properties of solutions. It identifies the dominant features and orders the approximates to the solutions, with fine corrections introduced successively. We will use the technique to analyze the solutions of the complex Ginzburg-Landau equation (CGLe) in chapter 4.

In all this study, symmetry considerations and Lie group theory have been gaining more and more attention [199, 1, 189, 156]. Interestingly, sometimes the precise form of the differential equation is not important for interpreting desired properties of a dynamical system. All that we need to know is the symmetry group associated with the system [101, 14, 46]. The continuity condition of solutions imposes severe constraints on the global topology.Through the construction of certain topological invariants, much can be deduced about the structure of the phase space [43, 183, 221, 186, 97].

In dissipative chaotic systems, the complicated asymptotic dynamics takes place on a compact subset which is not a manifold but has a fractal structure [164, 167], called a strange invariant set (SIS). Various fractal dimensions are introduced to characterize the size and geometric layout of the SIS [74, 226], and Chaoticity of the dynamics is shown by the biggest Lyapunov exponent [107, 226]. Statistical behavior of solutions depends heavily on the ergodic or mixing property of the system, the discussion of which constitutes a lively research direction itself [231]. The statistical properties are also investigated by introducing linear opearators, such as the Perron-Frobenius operator, and by studying their spectra. We will see how the periodic orbit theory is used to calculate the spectra later on.

The above program is carried out analytically only in a small number of systems. In most cases, numerical calculations are needed to assist analysis. Nowadays, with increasing power of digital computers, it is possible to carry out direct numerical simulation of more and more complex nonlinear problems. Numerical computation has become an indispensable tool for the exploration and application of dynamical systems. The computer generated figures provide intuitive pictures and stimulate inspiration for developing analytical methods [281]. Furthermore, numerical calculation with controlled error is an important tool for computer-assisted proofs of theorems [182]. Numerical procedures are also used to produce approximate invariant manifolds [127, 85, 26] which are used as the framework for qualitative discussion of the long-time behavior. Apart from equilibrium points, the simplest and most easily obtainable objects of a chaotic system are unstable periodic orbits (UPOs).

1.2 Periodic orbits: theory and numerics

One of the earliest observations of a dynamical system made by human beings is the periodic motion, *e.g.*, the circular motion of the heavenly bodies. Next discovery was the quasi-periodic motion, motion in which an orbit never retraces itself exactly but is decomposable to a finite set of components with incommensurate frequencies. It was natural to think that the most general form of bounded motion is the quasi-periodic oneb but the discovery of chaos shattered that belief. The chaotic flow has infinitely many UPOs with incommensurate frequencies, and its sensitive dependence on initial conditions precludes the possibility of quasi-periodic motion, even with infinitely many frequencies. It is a new form of motion,

with a distinct nature. Today, it is believed that chaos is the rule in general dynamical systems, and considerable amount of research efforts are devoted to it.

Although the set of UPOs has a zero Lesbegue measure on a SIS, its central importance was realized already by Poincaré [205]. In the early sixties, in order to explain the generation of a SIS, Smale [234] proposed the horseshoe mechanism. Loosely speaking, the SIS results from local stretching and squeezing and global bending and squashing. A SIS is usually a fractal set which is self-similar and covered densely by UPOs. If it is compact, maximal, hyperbolic and indecomposable [107], a Markov partition exists and a finite symbolic set could be defined such that the dynamics on the SIS is topologically equivalent to a subshift of finite type in the symbol space, and every periodic sequence corresponds to a UPO [107]. In general, the SIS has a much more complicated structure and infinitely many symbols are needed to characterize the set and the dynamics.

1.2.0.1 Dynamics and UPOs

The creation or destruction of UPOs under parameter changes indicates bifurcations. How is a periodic orbit generated? In general, there are two ways - through a local bifurcation or through a global bifurcation. Locally, a Hopf bifurcation is the commonly encountered way of producing a periodic orbit. When a complex pair of eigenvalues of the Jacobian matrix of a fixed point cross transversely the imaginary axis, Hopf bifurcation takes place, and the resulting periodic orbit inherits the stability of the fixed point [107]. Globally, a homoclinic bifurcation is frequently associated with appearance or disappearance of UPOs. Horseshoes are often generated by a homoclinic explosion like in the Lorenz system [236, 165], a special form of the homoclinic bifurcation, which results in a infinity of UPOs. Global saddle-node or pitchfork bifurcation are also common for the annhilation or creation of one or two UPOs.

The UPOs form the skeleton of a SIS [51, 50]. The closure of the set of UPOs is the SIS itself and the dynamics on it is a walk through the space of these UPOs. Various physically interesting quantities, like topological entropy, fractal dimension, correlation length and escape rate, can be calculated via spectral determinants or dynamical zeta functions [51], expressed as a convergent sum over the set of UPOs. If a complete symbolic dynamics exists, cycle expansions [9] give exponential (or super-exponential) convergence rate relative to the topological lengths of the UPOs.

We can give a geometrical picture of the role played by the UPOs in the SIS. Every UPO has a linearized neighborhood in which the dynamics is well approximated. Phase space is then partitioned into these neighborhoods. The more UPOs we have, the finer is the partition, the more accurate is the representation. In a hyperbolic system with complete symbolic dynamics, long orbits are shadowed by short ones, leaving only fast decaying curvature contributions. Cycle expansions on spectral determinants (dynamical zeta functions) are a systematic way of piecing the information together for calculating dynamical averages on a SIS.

1.2.0.2 UPOs and numerics

turbulence!hist

The very existence of UPOs tells us much about the dynamics. Sometimes, several shortest UPOs are enough to set up a frame for discussing qualitive properties of a dynamical system [51, 143]. If all the UPOs up to a given length are found, we may consider using trace formula to evaluate physical averages. It is, therefore, a crucial part of the periodic orbit theory to locate the requested UPOs with given accuracy. Analytical methods have very limited use in this context, so numerical integration is the only choice. Due to the sensitive dependence of solutions on initial conditions in a chaotic dynamical system, it is impossible to follow a typical orbit for arbitrarily long time with reasonable accuracy. To locate a UPO, however, requires only calculation on a finite orbit segment, which can be done with great accuracy [266]. If a UPO is hyperbolic, then it is the maximal invariant set in a small neighborhood of itself and persists under small perturbations, being either small changes of system parameters or numerical noise. So, a hyperbolic UPO is robust and numerically obtainable.

We are mainly concerned with UPOs on a SIS, and many different numerical methods have been developed to locate them. Basically, finding periodic orbits is a boundary value problem on the time axis. If a Poincaré section is chosen in phase space, simple shooting is often used for searching the shortest UPOs. In very chaotic system or to locate long UPOs, multipoint shooting is more stable [51, 242]. Variational methods [158, 150, 120] are useful for complicated or high-dimensional systems, which put a guessed loop in the phase space and then drive it to the desired periodic orbit by some extremization principle. To find periodic orbits in the discrete time series data from an experiment, time series analysis and embeddings are needed. In application of all these methods, it is very important to establish symbolic dynamics. In many systems, short UPOs can be found directly and used to establish symbolic dynamics which gives an upper bound of the number of the UPOs with a given length and helps predict the location of the longer orbits. Ideally no orbit is missing or over-counting. The construction is always possible if a Poincaré map of the SIS is hyperbolic.

1.3 Turbulence:brief notes

1.3.1 Historical perspective

Turbulence has long been one of the natural phenomena which receive most attention of the humankind. From soft breezes to violent hurricane, from boiling water to tsunami, from colorful chemical oscillation to magnificent solar magnetic explosion, everywhere, turbulence is present and has great influence on our life. It is among the most studied subjects but still quite poorly understood because of its extremely rich and complicated behavior, even though the underlying natural laws are clear. Navier-Stokes equation, for example, is a sufficiently good model for the fluid motion in many cases, and is regarded as containing necessary ingredients to explain most turbulent behavior of fluids. But lack of mathematical tools prevents us from fully understanding the nature of its solutions.

turbulence!rou to

With the accumulation of mathematical techniques and experimental facts, study on turbulence made steady progress, though somewhat slowly. Twilight shone from Kolmogorov's work in 1941 [148, 89], a classic in establishing phenomelogical models of turbulence. In addition to the Navier-Stokes equation, two universal assumptions were proposed to explain the experimentally observed two-thirds law and energy dissipation law. Landau questioned the universality and later proposed that turbulence is generated by sequence of Hopf bifurcations which activate infinitely many degrees of freedom with incommensurate frequencies [160]. Although the proposal was known to be incorrect later, Landau's emphasis on the investigation of the origin of turbulence is appreciated by many scientists.

After Lorenz's rediscovery of chaos in 1963, the stochastic motion in deterministic systems was recognized and studied. In 1971, Ruelle and Takens showed that quasi-periodic motion is not generic in general vector fields and the chaotic (strange) attractors occur more often than the high-dimensional tori. The argument was strong and thus made many people believe that chaotic dynamics is somehow closely related to the creation of turbulence. After that, different routes to turbulence were proposed and tested [89, 117, 49].

Experimentally, well-controlled fluid systems were devised and studied to check the various instabilities that lead to sequentially more complex motion. The discovery of coherent structures claimed the inadequacy of statistical description. Large-scale correlation is an significant part of turbulent motion in real fluids. Theoretically, models of various complexity are constructed and investigated in order to account for experimental observations and to extract essential elements of turbulence dynamics.

Nowadays, turbulence has become a general concept, familiar to many scientists in different areas. If a system possesses infinitely many degrees of freedom and the couplings between them are nonlinear, turbulence is likely to occur when control parameters cross certain critical values. Chemical turbulence, acoustic turbulence, optical turbulence and so on are named and studied, and new experimental techniques continue to be developed to test theoretical models and provide facts of turbulence. In extended systems, the study of turbulence is just as important as the study of regular motion due to its universality and genericity.

1.3.2 Main lines of approach

Kolmogorov's statistical model incorporates the symmetry consideration of the fluid system and is based on a multi-scale analysis: the average flow induces pulsations with largest size. Energy is transmitted to smaller and smaller scales in a hierarchical order via nonlinear interaction between modes of different scales. As a consequence, Small pulsations move in the background created by larger ones and have a smaller characteristic time. On the other hand, small scale motions induce extra dissipation and diffusion in an averaged way to the motion in large scales. The inspiration and the ideas contained in this model still influence current turbulence research [235, 90].

The ever improving experimental techniques keep revealing new quantitative features of turbulence which is a source of puzzles as well as inspirations. More subtle modeling is needed to give detailed description of turbulence and explain the observed experimental facts, like various coherent structures. Recently Numerical simulations achieve considerable success as the digital computer acquires stunning computing power. It is possible to directly evolve Navier-Stokes equation and do various manipulations that cannot be done in an experiment. But physical insights and physical consideration are still needed to make sense of the numerical calculation.

On the other hand, the development of dynamical system theory opened the door for establishing rigorous tools for investigating turbulence. In a finite system, At the onset of spatiotemporal complexity, a series of low-dimensional bifurcations take place which destroy the symmetries of the system steadily [125]. With more and more modes aroused, the spatial structure of the system becomes more and more complicated until the system reaches the spatiotemporal chaotic regime where the dynamics can be roughly viewed as a replica of building blocks juxposed to pave the spatial stretch and all the symmetries are restored in a statistical sense.

Various reduction schemes are used in the dymcial systems approach. P. Holmes *et al.* noticed that the spatial interaction is local and try to build local models to account for the coherent structures [125]. From the consideration of different space time scales, much can be achieved through the construction of amplitude or phase equations [260, 46, 190, 154]. About half a century ago, E. Hopf remarked that the turbulence motion in a finite space region really concentrates on a finite-dimensional manifold in the infinite-dimensional phase space of an extended system [130]. To embrace the idea, the concept of inertial manild is proposed and applied as a extension of the center manifold. On this manifold, the original partial differential equation (PDE) is equivalent to a finite set of ordinary differential equations (ODE).

Most theoretical investigation is based on the NSe which is a sufficiently good model of real fluid turbulence. The NSe has a bunch of symmetries [89], including space and time translational invariance, reflection and scaling invariance. If any model of turbulence is proposed, these symmetries should in a way be accomodated. Except for possible external force, a fluid particle receives action from its neighbors only, which means the coupling term in the NSe has a local nature. All the couplings between fluid particles have the same functional form and are the source of turbulent motion. So, as a dynamical system the NSe has many specific features. Similar is true for most other spatially extended systems. Their solutions should therefore exhibit these internal features. The coherent structures observed in many experiments and numerical simulations are characteristic of solutions and should be explainable from the structures of equations of motion. We will consider the coherent structures both from their origin and from their roles in the formation of spatiotemporal chaos.

1.3.3 Spatiotemporal chaos

In previous exposition, we used the term "spatiotemporal chaos" without much discussion. In our understanding, it is a synonym to "turbulence" in the dynamical systems language. Like the basic notions "turbulence", "chaos" [63], there is no unanimously-accepted rigorous mathematical definition for "spatiotemporal chaos". We give some description and characterization of it below.

The familiar term "chaos" usually describes a specific temporal behavior of the system - sensitive dependence on the initial conditions. The quantitative indicator is the leading Lyapunov exponent λ of the system dynamics. If $\lambda > 0$, generically, two nearby states separate exponentially fast and the dynamics is most likely to be chaotic. In this regime, the time correlation on a typical orbit decays exponentially and the orbit has a broad Fourier spectrum. In spatially extended systems, in analogy to the temporal chaos, it is useful to introduce the concept of chaos for the space dimension to characterize the spatial complexity and irregularity of the system. In a $1-d$ system, a spatial profile is said to be chaotic if the spatial correlation function decays exponentially with distances and the profile contains a broad spectrum of wavenumbers. Two chaotic profiles can be similar to each locally while their difference increases exponentially along the spatial axis.

Though very similar, temporal and spatial chaos are separate concepts. In low-dimensional systems, only temporal chaos exists. In the Lorenz system, the spatial represenatation is highly simplified - only three modes are used. So the spatial pattern is highly coherent - if two profiles agree quite well locally, then they agree well in the whole spatial stretch. But the temporal chaos in Lorenz equation is famous [164] and well established [181]. On the other hand, the steady solution set of the Kuramoto-Sivashinsky equation (KSe), which will be discussed in Chapter 5, is extremely complicated. The equations that the steady solutions satisfy has a chaotic repeller with the space coordiante x being the independent variable. However, their time dependence is trivial - steady.

A spatiotemporally chaotic system has complex dynamics in time and highly disordered spatial profiles. However, it does not mean that the system is totally erratic. Very often, regular elements are readily found in the system in both the space and the time extension. Spatially, coherent structures can be identified, which are defined locally and exhibit characteristic patterns with simple short-time dynamics. The lack of coherence between coherent structures accounts for the irregularity in space. Temporally, typical dynamics and characteristic time scales are invariable occurence. For example, in Lorenz system, the motion is characterized by the rotations around the two stationary points and the trasitions between them.

Rigorously, spatiotemporal chaos only happens in systems with infinite spatial extension, like that temporal chaos only appear in the infinite time limit. In numerical calculations or laboratory experiments what we can deal with are finite systems. We still say that the system exhibits spatiotemporal chaos if it shows those distinc chaotic character in both pace and time. Temporally, Lyapunov exponent is still a good indicator. Spatially, If a large set of active modes are present so that the spatial profile is highly irregular, then a chaotic description is suitable. See in figure **??** the evolution of the KSe in a finite-sized box which is regarded as spatiotemporal chaotic in this sense.

1.4 Periodic orbit theory on spatiotemporal chaos

Practically, if the existence of solutions of the NSe is taken for granted, convenient function spaces can be chosen, e.g., the set of smooth periodic functions are good for our investigations. With the chosen basis, the NSe converts to a countable set of coupled ODEs. The task is then to study to what extent the finite(low)-dimensional dynamical system theory, can be applied to infinite-dimensional systems, what happens during the transition to a turbulent state, and how the dynamics of a small system is related to that of the corresponding large system.

There is strong indication that a finite-dimensional phase space is enough to give a complete description of the fluid dynamics. In either numerical simulations or experimental observations, every so often, we catch a glimse of familiar patterns in the ever-evolving turbulent systems. These patterns are the aforementioned coherent structures and more importantly the dynamics is recurrent like a walk through a finite set of spatial patterns. In low-dimensional chaotic system, a typical phase point follows a UPO for a while and then another one, then the next...In spatially extended systems, in addition to this temporal decomposition by periodic orbits, there is spatial decomposition by systems of various sizes with periodic boundary conditions. Periodic orbits in these systems may be viewed as coherent structures. They are periodic in both space and time. The evolution of the system is shadowed by these compact invariant objects. For finite resolution, a finite number of periodic orbits are sufficient to approximate the strange set and the asymptotic dynamics in the whole phase space.

The primary goal of this thesis is to identify the coherent structures and investigate their dynamics from a periodic orbit theory point of view. Analytically, we studied the first instability of the CGLe and proved the existence of modulated amplitude waves (MAWs) which constitute basic building blocks for later spatiotemporal dynamics. We analyzed the structure of steady solutions of the KSe, identified the most importane ones and emphasized their role in organing the phase space dynamics. Numerically, we devised a novel variational technique to locate all the short UPOs in a spatiotemporally chaotic system. Using the set of UPOs, we systematically reconstructed the strange set, deduced the symbolic dynamics on the chosen Poincaré sections on a finite interval with periodic boundary conditions. Though the strange set is usually nonhyperbolic, with infinite bifurcations taking place even for the slightest changes of parameter values, the coarse structure of the strange set changes continuously. It is well shadowed by the finite set of periodic orbits the study of which sheds light on the spatiotemporal chaos in large systems.

Until now, work has been concentrating on the simplest models which exhibit spatiotemporal chaos, the complex Ginzburg-Landau equation (CGLe) and Kuramoto-Sivashinsky equation (KSe). As a starting try, all the calculation done here may well be view as preliminary, but the techniques and intuition developed will hopefully be extended to real world systems like the NSe, or other field equations.