8: The Path Integral for Free Field Theory

Prerequisite: 3, 7

Our results for the harmonic oscillator can be straightforwardly generalized to a free field theory with Hamiltonian density

\[ H_0 = \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2. \]  

(184)

The dictionary we need is

\begin{align*}
q(t) &\longrightarrow \varphi(x, t) \quad \text{(classical field)} \\
Q(t) &\longrightarrow \varphi(x, t) \quad \text{(operator field)} \\
f(t) &\longrightarrow J(x, t) \quad \text{(classical source)}
\end{align*}

(185)

The distinction between the classical field \( \varphi(x) \) and the corresponding operator field should be clear from context.

To employ the \( \epsilon \) trick, we multiply \( H_0 \) by \( 1 - i \epsilon \). The results are equivalent to replacing \( m^2 \) in \( H_0 \) with \( m^2 - i \epsilon \). From now on, for notational simplicity, we will write \( m^2 \) when we really mean \( m^2 - i \epsilon \).

Let us write down the path integral (also called the functional integral) for our free field theory:

\[ Z_0(J) \equiv \langle 0 | 0 \rangle_J = \int \mathcal{D} \varphi \, e^{i \int d^4x [\mathcal{L}_0 + J \varphi]}, \]  

(186)

where

\[ \mathcal{L}_0 = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2. \]  

(187)

Note that when we say path integral, we now mean a path in the space of field configurations.
We can evaluate $Z_0(J)$ by mimicking what we did for the harmonic oscillator in section 7. We introduce four-dimensional Fourier transforms,

$$\tilde{\varphi}(k) = \int d^4 x \, e^{-i k x} \varphi(x), \quad \varphi(x) = \int \frac{d^4 k}{(2\pi)^4} \, e^{i k x} \tilde{\varphi}(k),$$

(188)

where $k x = -k^0 t + \mathbf{k} \cdot \mathbf{x}$, and $k^0$ is an integration variable. Then, starting with $S_0 = \int d^4 x \, [\mathcal{L}_0 + J \varphi]$, we get

$$S_0 = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \left[ -\tilde{\varphi}(k)(k^2 + m^2)\tilde{\varphi}(-k) + \tilde{J}(k)\tilde{\varphi}(-k) + \tilde{J}(-k)\tilde{\varphi}(k) \right],$$

(189)

where $k^2 = \mathbf{k}^2 - (k^0)^2$. We now change path integration variables to

$$\tilde{\chi}(k) = \tilde{\varphi}(k) - \frac{\tilde{J}(k)}{k^2 + m^2}. \quad \text{(190)}$$

Since this is merely a shift by a constant, we have $\mathcal{D} \varphi = \mathcal{D} \chi$. The action becomes

$$S_0 = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \left[ \tilde{J}(k)\tilde{\varphi}(-k) - \tilde{\chi}(k)(k^2 + m^2)\tilde{\chi}(-k) \right].$$

(191)

Just as for the harmonic oscillator, the integral over $\chi$ simply yields a factor of $Z_0(0) = \langle 0|0 \rangle_{J=0} = 1$. Therefore

$$Z_0(J) = \exp \left[ -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \, \tilde{J}(k)\tilde{\varphi}(-k) \right],$$

$$= \exp \left[ -\frac{i}{2} \int d^4 x \, d^4 x' \, J(x) \Delta(x - x') J(x') \right].$$

(192)

Here we have defined the Feynman propagator,

$$\Delta(x - x') = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{i k (x-x')}}{k^2 + m^2 - i\epsilon}.$$

(193)

The Feynman propagator is a Green’s function for the Klein-Gordon equation,

$$(-\partial_x^2 + m^2) \Delta(x - x') = \delta^4(x - x').$$

(194)
This can be seen directly by plugging eq. (193) into eq. (194) and then taking the $\epsilon \to 0$ limit. We can also evaluate $\Delta(x - x')$ explicitly by treating the $k_0$ integral on the right-hand side of eq. (193) as a contour integration in the complex $k_0$ plane, and then evaluating the contour integral via the residue theorem. The result is

$$
\Delta(x - x') = \int d\bar{k} e^{i\bar{k}(x-x') - i\omega|t-t'|} = i\theta(t-t') \int d\bar{k} e^{i\bar{k}(x-x')} + i\theta(t'-t) \int d\bar{k} e^{-i\bar{k}(x-x')},
$$

(195)

where $\theta(t)$ is the unit step function. The integral over $\bar{d}k$ can also be performed in terms of Bessel functions; see section 4.

Now, by analogy with the formula for the ground-state expectation value of a time-ordered product of operators for the harmonic oscillator, we have

$$
\langle 0|T\varphi(x_1)\ldots|0 \rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} \ldots Z_0(J)\big|_{J=0}.
$$

(196)

Using our explicit formula, eq. (192), we have

$$
\langle 0|T\varphi(x_1)\varphi(x_2)|0 \rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} \frac{1}{i} \frac{\delta}{\delta J(x_2)} Z_0(J)\big|_{J=0}
$$

$$
= \frac{1}{i} \frac{\delta}{\delta J(x_1)} \left[ \int d^4x' \Delta(x_2 - x') J(x') \right] Z_0(J)\big|_{J=0}
$$

$$
= \left[ \frac{i}{\pi} \Delta(x_2 - x_1) + \text{term with } J\text{'s} \right] Z_0(J)\big|_{J=0}
$$

$$
= \frac{i}{\pi} \Delta(x_2 - x_1).
$$

(197)

We can continue in this way to compute the ground-state expectation value of the time-ordered product of more $\varphi$’s. If the number of $\varphi$’s is odd, then there is always a left-over $J$ in the prefactor, and so the result is zero. If the number of $\varphi$’s is even, then we must pair up the functional derivatives in an appropriate way to get a nonzero result. Thus, for example,

$$
\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0 \rangle = \frac{1}{i^2} \left[ \Delta(x_1-x_2)\Delta(x_3-x_4)
$$

$$
+ \Delta(x_1-x_3)\Delta(x_2-x_4)
$$

$$
+ \Delta(x_1-x_4)\Delta(x_2-x_3) \right].
$$

(198)
More generally,

$$
\langle 0| T \varphi(x_1) \ldots \varphi(x_{2n}) | 0 \rangle = \frac{1}{i^n} \sum_{\text{pairings}} \Delta(x_{i_1} - x_{i_2}) \ldots \Delta(x_{i_{2n-1}} - x_{i_{2n}}) .
$$

(199)

This result is known as Wick’s theorem.

**Problems**

8.1) Starting with eq. (193), verify eq. (194).

8.2) Starting with eq. (193), verify eq. (195).

8.3) Use eq. (86), the commutation relations eq. (95), and \( a(k)|0\rangle = 0, \langle 0|a^\dagger(k) = 0 \) to verify the last line of eq. (197).

8.4) The retarded and advanced Green’s functions for the Klein-Gordon wave operator satisfy \( \Delta_{\text{ret}}(x - y) = 0 \) for \( x^0 \geq y^0 \) and \( \Delta_{\text{adv}}(x - y) = 0 \) for \( x^0 \leq y^0 \). Find the pole prescriptions on the right-hand side of eq. (193) that yield these Green’s functions.

8.5) Let \( Z_0(J) = \exp iW_0(J) \), and evaluate the real and imaginary parts of \( W_0(J) \).

8.6) Repeat the analysis of this section for the complex scalar field that was introduced in problem 3.3, and further studied in problem 5.1. Write your source term in the form \( J^\dagger \varphi + J \varphi^\dagger \), and find an explicit formula, analogous to eq. (192), for \( Z_0(J^\dagger, J) \). Write down the appropriate generalization of eq. (196), and use it to compute \( \langle 0| T \varphi(x_1) \varphi(x_2) | 0 \rangle, \langle 0| T \varphi^\dagger(x_1) \varphi(x_2) | 0 \rangle \), and \( \langle 0| T \varphi^\dagger(x_1) \varphi^\dagger(x_2) | 0 \rangle \). Then verify your results by using the method of problem 8.3. Finally, give the appropriate generalization of eq. (199).