Particular solutions of shallow-water equations over a non-flat surface

K.V. Karelsky, V.V. Papkov, A.S. Petrosyan*, D.V. Tsygankov

Space Research Institute of The Russian Academy of Sciences, Profsoyuznaya 84 / 32, Moscow 117810, Russia

Received 26 June 1999; received in revised form 10 April 2000; accepted 17 May 2000

Communicated by A.P. Fordy

Abstract

It is shown that the generalization of elementary solutions of the classical shallow water equations to the case of a non-flat surface is possible only for the class of underlying surfaces for which simple wave solutions exist, namely for slopes of constant inclination. The simple self-similar solutions for the shallow-water equations over slopes are obtained and the principal nonexistence of simple wave solutions was shown for other surfaces. It is shown that the characteristics of the equations over an oblique plane are branches of parabolas that have second-order contact with the characteristics of an appropriate system of shallow-water equations over the flat surface. As a consequence, shallow water physics on slopes is essentially different. The coordinate transformation that transforms the one-dimensional Saint–Venant system of equations to that for the classical shallow-water equations is found and sufficient conditions for the existence of this transformation are obtained. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 03.40.Gc; 04.20.Jb; 92.60.Dj

Keywords: Particular solution; The shallow water equations; Variable depth; Dilatation wave; Shock wave; Riemann invariant

1. Introduction

The simple self-similar solutions of hyperbolic systems are basic for the study of non-linear wave phenomena as they allow us to find exact solution of the initial discontinuity decay problem. This is analogous to the role of the gasodynamics [1]. Those solutions are a key component in the development of computational methods based on analytical solutions of the Riemann problem. In the present work, the particular solutions of the classical shallow water system over non-homogeneous underlying surface are found and studied. This set of equations is known as the Saint–Venant equations [2]. The well-known classical shallow water equations describing an incompressible heavy fluid flow with a free surface on flat plate coincide with those of a polytropic gas with the specific heat ratio equals to two. However finding discontinuous solutions is not straightforward due to a difference in jump conditions at the discontinuity surface, although the continuous cases [3]. The equations used in the present article are not the same as the gasdynamic ones and the solutions ob-
tained are not trivial for continuous solutions either. It is shown that the generalisation of the elementary solutions of the classical shallow water equations over a flat boundary to the case of non-homogeneous bottom surface is possible only for a single type of underlying surfaces for which the simple wave solution exists. This class of surfaces consists of uniform slopes. The non-existence of simple wave solutions for the general types of bottom is shown, and the main attention is then given to the solutions of the appropriate system of equations on uniform slopes. The characteristics of the shallow water equations on uniform slopes are shown to be branches of parabolas which are in second-order contact with those for the shallow water system on a flat plate. This allows us to estimate the time interval for which the solutions of the Saint–Venant equations over a slope can be approximately replaced by the solutions of the classical system and to estimate the magnitude of the deviation as a function of time. As a consequence, unlike the classical case, strong jumps have a parabolic trajectory and propagate with constant acceleration. It was found that the simple compression wave solutions could exist only for a limited time interval. The jump in the derivatives of such solutions inevitably leads to a jump in functions as in the case of the classical shallow water equations. A minimum time interval, during which two arbitrary characteristics of given family of a simple compression waves intersect (and thus solutions are transformed into the class of discontinuous functions) is obtained. Both estimates show the limits of applicability of classical shallow water physics to real flows. Analytical solutions obtained describe the new physics associated with the slopes. Special attention is given to the behaviour of similarity solutions that play a key role in numerical methods based on Riemann solvers, namely that finite-difference jump relations on the surface of discontinuity is independent of the angle of slope. The solutions obtained allow us to show the coordinate transformation that changes the one-dimensional Saint–Venant equations into the system of classical shallow-water equations, and sufficient conditions for the existence of this transformation is found. This transformation is effectively used in a subsequent paper to solve the Riemann problem for shallow water equations on slopes.

2. Riemann invariants for the shallow-water equations over an arbitrary surface

In the present section we will derive Riemann invariants for the one-dimensional Saint–Venant equations describing shallow-water flows over a non-homogeneous surface that is determined by the function \( z = z(x) \):

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} &= 0, \\
\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + gh \frac{\partial (h + z)}{\partial x} &= 0.
\end{align*}
\]

(2.1)

Subsequently, it is convenient to use the matrix form of (2.1):

\[
\frac{\partial h}{\partial t} + \begin{pmatrix} u & h \\ g & u \end{pmatrix} \times \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial g}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ -g \frac{\partial z}{\partial x} \end{pmatrix}
\]

(2.2)

Let \( A \) denote the matrix:

\[
\begin{pmatrix} u & h \\ g & u \end{pmatrix}
\]

(2.3)

The Eqs. (2.2) are hyperbolic in the broad sense if the matrix \( A \) has two independent real eigenvectors and distinct eigenvalues. So, in order to determine the type of system (2.2) we need to find the eigenvalues of the matrix \( A \):

\[
\det \begin{pmatrix} u - \lambda & h \\ g & u - \lambda \end{pmatrix} = (u - \lambda)^2 - gh
\]

(2.4)

and as a result:

\[
\lambda_{1,2} = u \pm \sqrt{gh}
\]

(2.5)

Introducing the traditional notation:

\[
c = \sqrt{gh}
\]

(2.6)

we obtain two different real eigenvalues and two appropriate eigenvectors for system (2.2):

\[
\zeta_1 = \begin{pmatrix} -c \\ h \end{pmatrix}, \frac{1}{2}
\]

(2.7)

\[
\zeta_2 = \begin{pmatrix} c \\ h \end{pmatrix}, \frac{1}{2}
\]

(2.8)
The presence of the term on the right-hand side of the second of (2.2) does not, of course, disturbs the hyperbolicity of the system of equations. Therefore, the further study of (2.2) will be more convenient if we write it in terms of its Riemann invariants. For this purpose, we initially transform the system (2.2) to its characteristic form by multiplying by the appropriate eigenvector (2.7) or (2.8).

Multiplying (2.2) by the left eigenvector \( e \), we have:

\[
- \frac{c}{h} \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial x}{\partial x} - c \frac{\partial h}{\partial x} \right) + \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{c^2}{h} \frac{\partial z}{\partial x},
\]

and, finally, we obtain:

\[
- \frac{c}{h} \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial x}{\partial x} - c \frac{\partial h}{\partial x} \right) + \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{c^2}{h} \frac{\partial z}{\partial x},
\]

In the same way, multiplying (2.2) by the right eigenvector \( e \), we obtain:

\[
\frac{c}{h} \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial x}{\partial x} - c \frac{\partial h}{\partial x} \right) + \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{c^2}{h} \frac{\partial z}{\partial x}.
\]

It is convenient to put \( h \) into the differentiation operators on the left-hand side for further transformations of (2.10) and (2.11). For this purpose we introduce the function \( \varphi(h) \) such that \( \frac{\partial h}{\partial x} = \frac{\partial z}{\partial x} \). It is clear that considering \( h \) as the function of \( \varphi \), we obtain:

\[
\frac{\partial h}{\partial t} = \frac{\partial h}{\partial \varphi} \frac{\partial \varphi}{\partial t}
\]

\[
\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \varphi} \frac{\partial \varphi}{\partial x}
\]

Therefore:

\[
\varphi(h) = \int \frac{c}{h} dh = \int \frac{\sqrt{gh}}{h} dh
\]

The integration in (2.14) gives us:

\[
\varphi(h) = 2\sqrt{gh}
\]

Let us replace \( u, h \) by \( r, s \) as was first done by Riemann for the equations of gas dynamic. The variables \( r \) and \( s \) are called Riemann invariants, because the form of the equations rewritten in these variables is invariant under the replacement of the independent variables. This replacement is defined by the transition from the Lagrangian coordinates to the Eulerian coordinates, and vice versa. Thus we write:

\[
r = u + \varphi(h), \quad s = u - \varphi(h)
\]

As a result, we get:

\[
u = \frac{r + s}{2}, h = \frac{(r - s)^2}{16g}
\]

Furthermore, transforming the formulas (2.10), (2.11), and combining appropriate terms, we obtain the following equation for \( \chi(x,t) \):

\[
\left( \frac{\partial r}{\partial t} + \frac{\partial s}{\partial t} \right) + \left( \frac{1}{r} + \frac{1}{s} \right) \left( \frac{\partial r}{\partial x} + \frac{\partial s}{\partial x} \right) - \frac{8g}{(r - s)}
\]

\[
\times \left( \frac{r - s}{8g} \left( \frac{\partial r}{\partial t} - \frac{\partial s}{\partial t} \right) + \left( \frac{1}{r} + \frac{1}{s} \right) \right)
\]

\[
- \frac{r - s}{8g} \left( \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} \right) = -2g \frac{\partial z}{\partial x}
\]

Hence

\[
\frac{1}{2} \left( \frac{\partial r}{\partial t} + \frac{\partial s}{\partial t} \right) + \left( \frac{1}{r} + \frac{1}{s} \right) \left( \frac{\partial r}{\partial x} + \frac{\partial s}{\partial x} \right) - \frac{\partial r}{\partial t}
\]

\[
+ \frac{\partial s}{\partial t} - \left( \frac{1}{r} + \frac{1}{s} \right) \left( \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} \right) = -g \frac{\partial z}{\partial x}
\]

After regrouping we find

\[
\frac{\partial s}{\partial t} + \left( \frac{1}{r} + \frac{1}{s} \right) \frac{\partial s}{\partial x} = -g \frac{\partial z}{\partial x}
\]
In the same way, we obtain for \( r(x,t) \)
\[
\left( \frac{\partial r}{\partial t} + \frac{\partial s}{\partial t} \right) + \left( \frac{3}{2} r + \frac{1}{2} s \right) \left( \frac{\partial r}{\partial x} + \frac{s}{\partial x} \right) = - \frac{8 g}{r - s} \\
\times \left( \frac{r - s}{8 g} \left( \frac{\partial r}{\partial t} - \frac{\partial s}{\partial t} \right) + \left( \frac{3}{2} r + \frac{1}{2} s \right) \right) \\
\times \left( \frac{r - s}{8 g} \left( \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} \right) \right) = - 2 g \frac{\partial z}{\partial x}
\]
Hence
\[
1 + \frac{\partial r}{\partial t} + \frac{\partial s}{\partial t} \left( \frac{3}{2} r + \frac{1}{2} s \right) \left( \frac{\partial r}{\partial x} + \frac{s}{\partial x} \right) - \frac{\partial r}{\partial t} \\
+ \frac{\partial s}{\partial t} \left( \frac{3}{2} r + \frac{1}{2} s \right) \left( \frac{\partial r}{\partial x} + \frac{s}{\partial x} \right) = - g \frac{\partial z}{\partial x}
\]
And as a result
\[
\frac{\partial s}{\partial t} + \left( \frac{3}{2} r + \frac{1}{2} s \right) \frac{\partial s}{\partial x} = - g \frac{\partial z}{\partial x} \tag{2.18b}
\]

3. Continuous solutions: traveling and centred Riemann waves

The Eqs. (2.18a), (2.18b) differ in principle from the corresponding ones for the classical shallow-water model over a horizontal plane written in terms of the Riemann invariants. The function \(- g \partial z / \partial x\) on the right-hand side of the equations lead to the inapplicability of the classical Riemann wave solution in our case. By a definition accepted in gas dynamics, a Riemann wave (or a travelling wave) is a flow in which one of the Riemann invariants remains constant. One defines a backward Riemann wave when the \( r \)-invariant remains constant: \( r(x,t) = r(x_0,t_0) = r_0 \equiv \text{const.} \), and a forward Riemann wave, when the \( s \)-invariant remains constant: \( s(x,t) = s(x_0,t_0) = s_0 \equiv \text{const.} \). The equation for the \( r \)-invariant is automatically satisfied for the wave turned back, and the equation for the \( s \)-invariant is automatically satisfied in case of the wave turned forward. However, these definitions are improper for the systems (2.18a), (2.18b).

We define a Riemann wave turned back as a solution, which satisfies (2.18a) identically. Likewise, we define a Riemann wave turned forward as a solution, which satisfies the Eq. (2.18b) identically.

The reason for these definitions will be seen below. Let us find an expression for \( r(x,t) \), that satisfies Eq. (2.18b) identically in some range of \((x,t)\). For this purpose it is necessary that the coefficient of \( (\frac{3}{2} r + \frac{1}{2} s) \) is zero, because \( r(x,t) \) and \( s(x,t) \) are linearly independent functions. As a consequence \( \partial r / \partial x = 0 \). However it is clear that the Eq. (1.18b) cannot be satisfied if \( \partial r / \partial t \equiv 0 \). Thus, \( r(x,t) \) is a time dependent function, i.e. \( r = r(t) \). As a result, \( \partial / \partial x (\partial r / \partial t) \equiv 0 \) and therefore \( \partial / \partial x (- g \partial z / \partial x) \equiv 0 \). Hence the solution, which satisfies the Eq. (2.18a) identically, can exist only if the underlying surface is determined by the function \( z(x) \) that satisfies the equation \( \partial^2 z / \partial x^2(z) \), i.e. \( z = k x + z_0 \). The simple Riemann wave solution does not exist for other underlying surfaces.

Therefore hereafter we will consider \( \partial z / \partial x = k \equiv \text{const.} \). So,
\[
r = - g k t + r_0 \tag{3.1}
\]
satisfies (2.18b) identically. Substituting (3.1) into the Eq. (2.18a), we obtain:
\[
\frac{\partial s}{\partial t} + \left( \frac{3}{2} s - \frac{1}{2} g k t + \frac{1}{2} r_0 \right) \frac{\partial s}{\partial x} = - g k \tag{3.2}
\]
It is obvious, that along the integral curves \( X(t) = \text{const.} \), which are called characteristics, and satisfy the equation:
\[
\frac{dx}{dt} = u - c = \frac{3}{2} s - \frac{1}{2} g k t + \frac{1}{2} r_0, \quad \tag{3.3}
\]
the Eq. (3.2) takes the form:
\[
\frac{\partial s}{\partial t} + \frac{dx}{dt} \frac{\partial s}{\partial x} = - g k \Leftrightarrow \frac{ds}{dt} = - g k \tag{3.4}
\]
Integrating (3.4), we have:
\[
s(X(t),t) = \int_0^t \frac{ds}{dt} dt = - g k t + s(X(0),0) \tag{3.5}
\]
Now, substituting (3.5) into (3.3), we get:
\[
\frac{dx}{dt} = \frac{1}{4} (3 g k t + 3 s(X(0),0) - 3 g k t + \frac{1}{2} r_0) \\
= - g k t + \frac{3 s(X(0),0) + r_0}{4} \tag{3.6}
\]
After integration of (3.6), we obtain an expression for \( X(t) \) in explicit form:

\[
X(t) = \int_0^t \frac{dx}{dt} dt = -\frac{1}{2} gkt^2 + \frac{3s(X(0),0)}{4} \frac{t}{X(0)} + X(0)
\]

(3.7)

Consequently, the integral curves (3.7) are parabolas in the \((x,t)\) plane. The solutions of the shallow-water equations on slopes differ from the solutions of the classical shallow-water equations by the term \(-\frac{1}{2} gkt^2\). This enables us to estimate the accuracy of approximate solution of the Saint-Venant equations to those of the classical equations as a function of time. Similar relations, except for notation, are obtained for the Riemann wave turned forward:

\[
s = -gkt + s_0;
\]

\[
r(X(t),t) = -gkt + r(X(0),0) \quad (2.8.2.9)
\]

\[
X(t) = -\frac{1}{2} gkt^2 + \frac{3r(X(0),0)}{4} \frac{t}{X(0)} + X(0)
\]

(3.10)

If \( \partial s/\partial x > 0 \) in some domain \((x,t)\) for the Riemann wave turned back, the integral curves (3.10) are divergent, and, taking into account (2.17), we get \( u = \frac{1}{2}(r_0 + s) \). As a result \( \partial u/\partial x = \frac{1}{2} \partial s/\partial x \) \( \partial u/\partial x > 0 \), and, considering mass conservation for the system (2.1) we have: \( \partial h/\partial x < 0 \), i.e. \( h \) decreases. Hence, we have a rarefaction wave. If \( \partial s/\partial x < 0 \) in some domain \((x,t)\), the integral curves (3.10) are convergent, and we have a compression wave. In the domain \((x,t)\), in which \( \partial s/\partial x = 0 \), the characteristics are parallel lines, and we have a zone of uniformly accelerated flow.

The same results (to within a sign) are obtained for a wave turned forward \((s(x,t) = s_0 = \text{const.})\). If \( \partial r/\partial x < 0 \), we have a rarefaction wave; if \( \partial r/\partial x > 0 \), we have a compression wave, and if \( \partial r/\partial x = 0 \), we have a zone of uniformly accelerated flow.

Using the expressions (2.18a), (2.18b) we will get relations for simple waves. So, for the backward Riemann wave we obtain the following result:

\[
\begin{cases}
u(x,t) + 2c(x,t) - gzt = u_0 + 2c_0, \\
u(x,t) - 2c(x,t) - gzt = u(x,0) - 2c(x,0)
\end{cases}
\]

\[
\frac{dx}{dt} = \frac{3u(x,0) + u_0}{4} + \frac{3c(x,0) - c_0}{2} - gkt
\]

(3.6a)

For the case of the forward Riemann wave we have the result:

\[
\begin{cases}
u(x,t) - 2c(x,t) - gzt = u_0 - 2c_0, \\
u(x,t) + 2c(x,t) - gzt = u(x,0) + 2c(x,0)
\end{cases}
\]

\[
\frac{dx}{dt} = \frac{3u(x,0) + u_0}{4} + \frac{3c(x,0) - c_0}{2} - gkt
\]

(3.6b)

We consider next the practically important special case of a travelling wave. The backward Riemann wave \((r(x,t) = r_0 = \text{const.})\) is termed centred wave, if the characteristics (3.1) form a group of curves that come out of one point \((\bar{x},\bar{t})\). Let us denote the parameter, which can assume all values from the segment \([s(\bar{x} - 0,\bar{t}),s(\bar{x} + 0,\bar{t})]\), by \( \bar{s} \). For the sake of simplicity suppose \( \bar{t} = 0, \bar{x} = 0 \). Then the solution is determined by the conditions:

\[
\begin{cases}
u(x,t) + 2c(x,t) - gzt = u_0 + 2c_0, \\
u(x,t) - 2c(x,t) - gzt = u_0 - 2c_0
\end{cases}
\]

\[
x = \frac{3\bar{x} + u_0 + 2c_0}{4} \frac{t}{1 - \frac{1}{2} gkt^2}
\]

(3.7a)

In the same way for a centred forward Riemann wave turned forward \((s(x,t) = s_0 = \text{const.})\), we denote the parameter, which can assume all values from the segment \([r(\bar{x} - 0,\bar{t}),r(\bar{x} + 0,\bar{t})]\), by \( \bar{r} \) and obtain:

\[
\begin{cases}
u(x,t) - 2c(x,t) - gzt = u_0 - 2c_0, \\
u(x,t) + 2c(x,t) - gzt = u_0 + 2c_0
\end{cases}
\]

\[
x = \frac{3\bar{x} + u_0 - 2c_0}{4} \frac{t}{1 - \frac{1}{2} gkt^2}
\]

(3.7b)

It is necessary to stress that, as is clear from relations (3.7a), (3.7b), the Eqs. (2.1) do not have continuous similarity solutions. Thus, the simple compression wave solution can exist only during a limited time as
in the classical case. It is because the characteristics of the appropriate set for a compression wave converge in time; and the absolute values of the gradients $\partial u/\partial x, \partial h/\partial x$ increase, until $t = t_0$, when the characteristics intersect, and, consequently, the gradients become unrestricted. The time interval $\Delta t$, during which the characteristics $X_i(t), X_i(t)$ of a $r$-compression wave intersect is:

$$\Delta t = \frac{4}{3} \frac{(X_2(0) - X_1(0))}{u(X_1(0),0) - u(X_2(0),0) - 2(c(X_1(0),0) - c(X_1(0),0))}. \quad (3.10a)$$

Using (3.10a), we can easily calculate the time interval of the existence of a continuous solution $\Delta t_{in}$:

$$\Delta t_{min} = \min_{x_1,x_2} \{\Delta t\} = -\frac{4}{3} \max x \left\{ \frac{1}{\frac{\partial u}{\partial x}} \left[ \frac{\partial c}{\partial x} \right] \right\} \quad (3.11a)$$

Similar relations for an $s$ compression wave are

$$\Delta t = \frac{4}{3} \frac{(X_2(0) - X_1(0))}{u(X_1(0),0) - u(X_2(0),0) + 2(c(X_1(0),0) - c(X_1(0),0))}. \quad (3.10b)$$

$$\Delta t_{min} = \min_{x_1,x_2} \{\Delta t\} = -\frac{4}{3} \max x \left\{ \frac{1}{\frac{\partial u}{\partial x}} \left[ \frac{\partial c}{\partial x} \right] \right\} \quad (3.11b)$$

Thus, after the expiry of the time $\Delta t_{min}$, the compression wave transforms to discontinuous solutions. As is known from hyperbolic systems theory, the envelope curve of the intersections of the characteristics for slightly nonlinear systems, to which the system (2.1) belongs, is also a characteristic.

4. Discontinuous solutions. The jump conditions

In this section the conditions that must be observed on the discontinuity lines are obtained. For this purpose we rewrite the system (2.1) in the divergence form:

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0; \quad \frac{\partial hu}{\partial t} + \frac{\partial hu^2}{\partial x} + \frac{g}{2} \frac{\partial h^2}{\partial x} = - gh \frac{\partial z}{\partial x}. \quad (4.1)$$

Integrating (4.1) on an arbitrary domain $G$ that is homeomorphic to a square in the $(x,t)$ plane, we obtain:

$$\int \int \left( \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} \right) dG = 0; \quad \int \int \left( \frac{\partial hu}{\partial t} + \frac{\partial hu^2}{\partial x} \right) dG = - \int \int gh \frac{\partial z}{\partial x} dG. \quad (4.2)$$

Transforming the volume integrals on the left-hand sides of the system (4.2), using the Greens formula, we obtain:

$$\oint_{\partial G} (hu) dx - (hu) dt = 0; \quad \oint_{\partial G} (hu^2 + \frac{g}{2} h^2) dt = - \int \int gh \frac{\partial z}{\partial x} dG \quad (4.3)$$

Eqs. (4.3) represent the most general relations that are integral conservation laws and is valid for an arbitrary contour $\partial G$ and, in particular, for the contour which includes the discontinuity lines of an appropriate solution.

Let $x = x(t)$ be the equation of a jump-line and suppose it has a continuous tangent on the segment $[t_1,t_2]$. Since the functions $u(x,t), h(x,t)$ have a jump only on the line $x = x(t)$, we denote:

$$u_i(t) = \lim_{x \rightarrow x_i(t) -} u(x,t)u_2(t) = \lim_{x \rightarrow x_i(t) +} u(x,t); \quad h_i(t) = \lim_{x \rightarrow x_i(t) -} h(x,t)h_2(t) = \lim_{x \rightarrow x_i(t) +} h(x,t). \quad (4.4)$$
Let $\partial G$ be the contour ABCE, which has lines AB and CE that are located infinitely close about to the line of the jump $x(t)$ on the left and on the right sides respectively (see Fig. 1). Putting in consideration the speed of discontinuity $D = D(t) = x'(t)$, i.e. $dx = D(t)dt$, we have:

$$
\left\{ \begin{array}{l}
\phi_{AB} (Dh - hu) dt - \phi_{CE} (Dh - hu) dt = 0 \\
\phi_{AB} (Dhu - \left( hu^2 + \frac{g}{2} h^2 \right)) dt \\
- \phi_{CE} (Dhu - \left( hu^2 + \frac{g}{2} h^2 \right)) dt \\
= - \int gh \frac{\partial z}{\partial x} dG 
\end{array} \right.
$$

(4.5)

Because ABCE is an arbitrary contour, the relations (4.5) are equivalent to the following conditions for the integrands:

$$
Dh_1 - h_1 u_1 = Dh_2 - h_2 u_2;
$$

$$
Dh_1 u_1 - \left( h_1 u_1^2 + \frac{g}{2} h_1^2 \right) = Dh_2 u_2 - \left( h_2 u_2^2 + \frac{g}{2} h_2^2 \right).
$$

(4.6)

Now, using the operator $[ ]$ to denote the jumps in $u(x,t), h(x,t)$ on the discontinuity $x = x(t)$ and, using the relations (4.4), we get:

$$
[u] = u_2(t) - u_1(t)
$$

$$
[h] = h_2(t) - h_1(t)
$$

(4.7)

The general form of the jump conditions will be:

$$
D[h] = [hu]
$$

$$
D[hu] = \left[ hu^2 + \frac{g}{2} h^2 \right]
$$

(4.8)

Hence, the jump conditions (4.8) coincide with the Hugoniot conditions for the shallow-water over a horizontal surface. After simple transformations we obtain the relationship between the values of the main functions and the speed of a jump:

$$
D = \frac{h_1 u_1 - h_2 u_2}{h_1 - h_2},
$$

$$
\left( h_1 u_1 - h_2 u_2 \right)^2 - \left( h_1 u_1^2 - h_2 u_2^2 \right) (h_1 - h_2)
$$

$$
= \frac{g}{2} (h_1^2 - h_2^2).
$$

(4.9)

As a result we find:

$$
u_1 = u_2 - (h_1 - h_2) \sqrt{\frac{g}{2} \frac{h_1 + h_2}{h_1 h_2}}
$$

(4.10)

Thus, the conditions on a strong jump in our case coincide identically with the conditions on a jump for a classical system of shallow-water equations. However, unlike the classical case, the strong jumps have a parabolic trajectory and propagate with the constant acceleration $-gk$.

5. The transformation of the Saint–Venant equations to the classical shallow-water equations

Analysis of the results obtained in Sections 3 and 4 enables us to find an explicit nondegenerate replacement of the dependent and independent variables that transform (2.1) to the classical equations describing the shallow-water flows on a flat surface. Assuming, as above, that $k = \partial z/\partial x =$ const. we rewrite (2.1) in the form:

$$
\left\{ \begin{array}{l}
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0; \\
\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} + u \frac{\partial u}{\partial x} = -gk
\end{array} \right.
$$

(5.1)

and make the following replacement of the variables:

$$
\left\{ \begin{array}{l}
\tilde{x} = x + \frac{1}{2} gkt^2; \\
\tilde{t} = t
\end{array} \right.
$$

(5.2)
Thus,
\[
\begin{align*}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial t} + gk\frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x}
\end{align*}
\]  
(5.3)

It is clear that the transformation (5.2) is nondegenerate, because it has the positive Jacobian:
\[
J = \det \begin{vmatrix}
\frac{\partial t}{\partial\tilde{t}} & \frac{\partial t}{\partial\tilde{x}} \\
\frac{\partial x}{\partial\tilde{t}} & \frac{\partial x}{\partial\tilde{x}}
\end{vmatrix} = \det \begin{vmatrix}
1 & 0 \\
gk & 1
\end{vmatrix} = 1
\]  
(5.4)

Hence, applying (5.3) to (5.1), we have:
\[
\begin{align*}
\frac{\partial h}{\partial\tilde{t}} + (u + gk\tilde{t}) \frac{\partial h}{\partial\tilde{x}} + \frac{u}{\partial\tilde{x}} &= 0; \\
\frac{\partial u}{\partial\tilde{t}} + gk \frac{\partial h}{\partial\tilde{x}} + (u + gk\tilde{t}) \frac{\partial u}{\partial\tilde{x}} &= -gk
\end{align*}
\]  
(5.5)

Combining the appropriate terms in (5.5), we obtain:
\[
\begin{align*}
\frac{\partial h}{\partial\tilde{t}} + (u + gk\tilde{t}) \frac{\partial h}{\partial\tilde{x}} + \frac{\partial(u + gk\tilde{t})}{\partial\tilde{x}} &= 0; \\
\frac{\partial(u + gk\tilde{t})}{\partial\tilde{t}} + gk \frac{\partial h}{\partial\tilde{x}} + (u + gk\tilde{t}) \frac{\partial(u + gk\tilde{t})}{\partial\tilde{x}} &= 0
\end{align*}
\]  
(5.6)

Making replacement \(u = u - gk\tilde{t}\) in (5.6), we obtain the classical system of the shallow-water equations:
\[
\begin{align*}
\frac{\partial h}{\partial\tilde{t}} + \frac{\partial h}{\partial\tilde{x}} + \frac{\partial u}{\partial\tilde{x}} &= 0; \\
\frac{\partial u}{\partial\tilde{t}} + gk \frac{\partial h}{\partial\tilde{x}} + \frac{\partial u}{\partial\tilde{x}} &= 0
\end{align*}
\]  
(5.7)

This replacement is equivalent to the transition to a non-inertial reference frame that moves with the constant acceleration \(gk\) parallel to the \(x\) axes. If \(k \neq \text{const.}\), such a replacement is generally impossible because of additional dynamic forces, which make the Jacobian (5.4) to be degenerate. Thus, when the underlying surface is a slope, transition to the appropriate non-inertial reference frame (i.e. the replacement of coordinates) permits us to remove the free term in the one-dimensional Saint–Venant Eqs. (2.1) and to operate with the usual system of shallow-water Eqs. (5.7).

6. Conclusion

In the present work the generalization of classical shallow-water theory to the case of flows over a non-homogeneous underlying surface is executed. It is shown that the simple selfsimilar solutions that are characteristic for the classical problem exist only if the underlying surface is a uniform slope. The particular solutions of the shallow-water equations over this slope are obtained and the features of these solutions (differences from the solutions of the classical equations) are analyzed. Study of these solutions shows that it is not possible for the impenetrability conditions on the solid wall to hold for the boundary problem of shallow-water flow over a slope.

The particular solutions obtained enable us to present the nondegenerate replacement of the dependent and independent variables that transforms the Saint–Venant equations for a slope to the classical shallow-water ones. This replacement provides the effective consideration of the initial discontinuity decay problem on slopes. The solution of this problem will be given in the next paper.

References