2-d ISING MODEL ON TRIANGULAR LATTICE

Short version: Implement the real-space renormalization on the Ising model on a 2-dimensional triangular lattice.

Purpose: Learn a systematic real-space renormalization technique.

Guided tour version:

Triangular lattice:

\[ \mathcal{H} = K \sum_{\langle ij \rangle} S_i S_j + h \sum_i S_i \]

\[ S_i = \{ -1, +1 \} \text{, a block } \sigma_I = \{ S_1^x, S_2^x, S_3^x \} \]

Block spin by majority rule

\[ S_I = \text{sign} \{ S_1^x + S_2^x + S_3^x \} \]

Lattice constant \( a \rightarrow a\sqrt{3} \).

Write down the table of 8 distinct \( \sigma_1, \sigma_2, \sigma_3 \) configurations:

\[ S_1 = +1 \leftarrow \uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow \]

\[ S_1 = -1 \leftarrow \downarrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \uparrow\uparrow \]

for later use (9.148)

(9.149)
Coarse grained Hamiltonian:

\[ e^{\mathcal{H}^{[S_1]}} = \sum_{\{\sigma_i\}} e^{\mathcal{H}^{[S_1,\sigma_i]}} \quad (9.150) \]

We concentrate on \( h = 0 \), no external field case.

\[ \mathcal{H} = \mathcal{H}_0 + V \]

\[ \downarrow \quad \text{within the block} \quad \quad \quad \rightarrow \text{across the blocks} \]

\[ \mathcal{H}_0 = k \sum_i \sum_{j \in I} S_i S_j \quad V = K \sum_{i \neq j} \sum_{j \in J} S_i S_j \]

We will develop perturbation theory using \( \mathcal{H}_0 \) as the "nonperturbed" Hamiltonian, with averages defined as

\[ \langle A \rangle_0 = \frac{\sum_{\{\sigma_i\}} e^{\mathcal{H}_0^{[S_1,\sigma_i]}} A^{[S_i,\sigma_i]} \sum_{\{\sigma_i\}} e^{\mathcal{H}_0^{[S_1,\sigma_i]}}}{\sum_{\{\sigma_i\}} e^{\mathcal{H}_0^{[S_1,\sigma_i]}}} \]

\[ \Theta \text{ Show that} \]

\[ \sum_{\{\sigma_i\}} e^{\mathcal{H}_0^{[S_i,\sigma_i]}} = Z_0(k)^M \]

\[ Z_0(k) = e^{3k} + 3e^{-k} \]

\[ \Theta \text{ Does this result depend on the block-spin } S_1? \]
Perturbation theory

Show that

$$\langle e^V \rangle_0 = e^{\langle V \rangle_0 + \frac{1}{2}(\langle V^2 \rangle_0 - \langle V \rangle_0^2)} + \cdots$$

$$\mathcal{H}^{I}[S_{1}] = M \ln Z_0(K) + \langle V \rangle_0 + \frac{1}{2}(\langle V^2 \rangle_0 - \langle V \rangle_0^2) + \cdots$$

First order in $V$

$$V = \sum_{I \neq J} V_{IJ}$$

$$V_{IJ} = K S^I_3 (S^I_1 + S^I_2)$$

$$\langle V_{IJ} \rangle_0 = 2K \langle S^I_3 S^I_1 \rangle_0$$

Show that

$$\langle V_{IJ} \rangle_0 = 2K \langle S^I_3 \rangle_0 \langle S^I_1 \rangle_0$$

$$\langle S^I_3 \rangle = S^I_3 \Phi(K), \quad \Phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

$$\mathcal{H}^{I}[S_{1}] = M \ln Z_0(K) + K' \sum_{I \neq J} S^I_3 S^I_1 + O(v^2)$$

$$K' = R(K) = 2K \Phi(K)^2$$
Fixed points, critical exponents

show that the fixed points

\[ K^* = 2K^* \Phi(K^*)^2 \]

are

\[ K_c = \frac{1}{4} \ln (1 + 2\sqrt{2}) \]

how does that compare to Onsager's exact result \( K_c = 0.27 \ldots \)?

show that the eigenvalue

\[ \Lambda_1 = \frac{\partial \xi_c(K)}{\partial K} \bigg|_{K_c} \approx 1.81 \ldots \]  
\[ \approx 1.62 \ldots \]  
\[ -0.27 \ldots \]  

(Onsager says \( \Lambda_1 = \sqrt{3} \))

discuss the behavior of correlation length at \( K_c \)
2nd order in $V$

\[
\langle V^2 \rangle_0 - \langle V \rangle_0^2 = K \sum_{i,j} \sum_{m,n} \left( \langle S_i S_j S_m S_n \rangle_0 - \langle S_i S_j \rangle_0 \langle S_m S_n \rangle_0 \right)
\]

This you already know.

(*) Show that $\langle V^2 \rangle_0 - \langle V \rangle_0^2 = 0$ unless

- $S_i, S_m$ in the same block
- $S_j, S_n$ in the same block

(or other way around)

Forces 2nd and 3rd nearest neighbor interactions!

(If you get next part done, I'll be impressed):

(*) Show that in terms of 1st, 2nd and 3rd neighbor interactions there are 3 eigenvalues

\[
\Lambda_1 = 1.77...
\]
\[
\Lambda_2 = 0.23...
\]
\[
\Lambda_3 = -0.12...
\]

What is relevant, what is irrelevant in this renormalized theory as $K \rightarrow K_c$? Are you closer to the exact Onsager result?

# end of exam #