Problem 3.2.6 (Eliminating the cubic term)

Consider the system
\[ \dot{X} = RX - X^2 + aX^3 + O(X^4), \]
where \( R \neq 0 \). We want to find a new variable \( x \) such that the system transforms into
\[ \dot{x} = Rx - x^2 + O(x^4). \]

This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped up to fourth order.

Let \( x = X + bX^3 + O(X^4) \), where \( b \) will be chosen later to eliminate the cubic term in the differential equation for \( x \). This is called a near-identity transformation, since \( x \) and \( X \) are practically equal; they differ by a tiny cubic term. (We have skipped the quadratic term \( X^2 \), because it is not needed—you should check this later.) Now we need to rewrite the system in terms of \( x \); this calculation requires a few steps.

(a) 
Show that the near-identity transformation can be inverted to yield \( X = x + cx^3 + O(x^4) \), and solve for \( c \).

(b) 
Write \( \dot{x} = \dot{X} + 3bX^2 \dot{X} + O(X^4) \), and substitute for \( X \) and \( \dot{X} \) on the right-hand side, so that everything depends only on \( x \). Multiply the resulting series expansions and collect terms, to obtain \( \dot{x} = Rx - x^2 + kx^3 + O(x^4) \), where \( k \) depends on \( a, b, \) and \( R \).

(c) 
Now the moment of triumph: choose \( b \) so that \( k = 0 \).

(d) 
Is it really necessary to make the assumption \( R \neq 0 \)? Explain.