0. The Calculus of Variations

0.1 Introduction to the calculus of variations: what the calculus of variations is good for; calculus of variations with one dependent and one independent variable, variations with fixed end-points, Euler’s equation; readily integrable systems, conservation laws

0.2 Several dependent and several independent variables: classical mechanics; the scalar wave equation; Maxwell’s equations; the Schrödinger equation

0.3 Variations at boundaries: natural boundary conditions; superconductivity

0.4 The second variation: stationarity versus extremality; the criteria of Legendre, Euler and Jacobi

0.4 Constraints: implementing constraints via Lagrange multipliers; isoperimetric problems; connection with eigenproblems; variational approximation schemes

1. Partial differential equations of mathematical physics

1.1 A selection of important partial differential equations (or PDEs): the diffusion equation (heat conduction, stochastic processes, polymer statistics); the wave equation (motion of transverse displacements of a stretched string); Maxwell’s equations (in vacuo) and potentials in electromagnetism; the time-dependent Schrödinger equation; the Navier-Stokes equation (motion of simple fluids); time-independent Ginzburg-Landau equation (equilibrium states of superconductors); time-independent Landau-Lifshitz equation (equilibrium states of hard magnets)

1.2 Classification of partial differential equations and boundary conditions: how to think about boundary conditions; cases of interest to us; general and special PDEs; the Cauchy-Kovalevska theorem; classification of quasi-linear second order PDEs; characteristic curves; qualitative features of characteristic curves

2. Separation of variables

2.1 Introduction

2.2 Separating the time dependence

2.3 Separable coordinates: rectangular, circular cylindrical and spherical polar

2.4 Series solutions of ordinary differential equations

2.5 Sturm-Liouville eigenproblems: matrix eigenproblems; linear operators, adjoint operators; boundary conditions, adjoint and self-adjoint; Sturm-Liouville form of the general eigenproblem; examples of Sturm-Liouville forms of eigenproblems; properties of Sturm-Liouville operators; Fourier series; Fourier transforms and Fourier integrals

3. Spherical harmonics and their applications

3.1 Construction of spherical harmonics: series solution of the associated Legendre equation; Legendre polynomials and their properties; associated Legendre functions and their properties
3.2 Spherical harmonic functions and their properties: addition theorem for spherical harmonics; multipole expansions

3.3 Laplace’s equation in spherical polar coordinates: uniqueness of solutions of the Laplace equation; interior Laplace problem; exterior Laplace problem; region between concentric spheres; neutral conducting sphere in a uniform external field

4. Bessel functions and their applications

4.1 Series solution of Bessel’s equation

4.2 Neumann functions

4.3 Some properties of solutions of Bessel’s equation

4.4 Bessel functions of imaginary argument

4.5 Laplace’s equation in cylindrical polar coordinates: application to fluid mechanics

4.6 Bessel series as analogues of Fourier series

4.7 Solution of Laplace’s equation inside an infinitely long cylinder

4.8 Bessel transforms as analogues of Fourier transforms

5. Normal mode eigenproblems

5.1 Separating the time-dependence

5.2 Diffusion equation in a closed region of space

5.3 Normal mode treatment of the drumhead

5.4 Normal mode treatment of the time dependent Schrödinger equation

5.5 Acoustic wave guides

6. Inhomogeneous ordinary differential equations

6.1 Introduction

6.2 Inhomogeneous ordinary differential equations: variation of parameters; Green functions (GFs) for inhomogeneous ordinary differential equations (ODEs); the issue of boundary conditions for GFs for ODEs

6.3 Equivalence of inverse matrices and GFs: eigenvector expansion for matrix inverses; reciprocity and its physical origin

6.4 Example – GF for inhomogeneous ODE: the bowed stretched string; inhomogeneity

6.5 Eigenfunction expansion for GF

7. Inhomogeneous partial differential equations and Green functions

7.1 Poisson’s equation: electrostatics in the presence of charge; solution using Green’s theorem; the basic GF – Coulomb’s law; Poisson’s equation when there is no boundary; Green function for Poisson’s equation inside a sphere; expansion of Green function in spherical polar coordinates; example – electrostatic potential inside a grounded conducting sphere with charges; how to compute a Green function when no images trick is apparent; Green function for Poisson’s equation for the interior of an infinite cylinder
7.2 Green functions and conversion of differential equations into integral equations

7.3 Green functions in the time-dependent domain: wave equation; boundary and initial conditions for the wave equation; how we would use the Green function if we knew it; source-varying Green function; Green function for wave equation in an infinite spatial region; use of the causal Green function; Liénard-Wiechert potential; computation and use of a source-varying Green function – example

8. Integral equations

8.1 Introduction: why we study integral equations; classification of linear integral equations

8.2 Integral transforms: review of some familiar integral transforms

8.3 Integral equations with degenerate kernels

8.4 The Fredholm alternative

8.5 Neumann series solution to integral equations

8.6 Fredholm’s formula: some properties of gaussian integrals; Wick’s theorem from gaussian integrals; complex gaussian integrals; Grassmann gaussian integrals

8.7 Fredholm’s method from Grassmann functional integrals: example of Fredholm’s method; Fredholm eigenproblem

9. Boundary integral methods

9.1 Single and double layer potentials

9.2 Jump conditions and boundary integral equations

9.3 Applications to spectral geometry: “Can one hear the shape of a drum?”